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A new AdS/CFT correspondence

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Abstract: We consider a geometric zero-radius limit for strings on $AdS_5 \times S^5$, where the Anti-de Sitter hyperboloid becomes the projective lightcone. In this limit the fifth dimension becomes non dynamical, yielding a different "holographic" interpretation than the usual "bulk to boundary" one. When quantized on the random lattice, the fifth coordinate acts as a new kind of Schwinger parameter, producing Feynman rules with normal propagators at the tree level: for example, in the bosonic case ordinary massless ϕ^4 theory is obtained. In the superstring case we obtain new, manifestly $\mathcal{N}=4$ supersymmetric rules for $\mathcal{N}=4$ super Yang-Mills. These gluons are also different from those of the usual AdS/CFT correspondence: they are the "partons" that make up the usual "hadrons" of the open and closed strings in the familiar QCD string picture. Thus, their coupling g_{YM} and rank N of the "color" gauge group are different from those of the "flavor" gauge group of the open string. As a result we obtain different perturbation expansions in radius, coupling, and $1/N$.

Keywords: 1/N Expansion, Superstrings and Heterotic Strings, AdS/CFT Correspondance, QCD.

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Contents

1. Outline

The AdS/CFT correspondence [1] derives a four-dimensional conformal field theory from type-IIB superstring theory by choosing as a vacuum five-dimensional Anti-de Sitter space, whose symmetry SO(4, 2) is the same as the four-dimensional conformal group. The Maldacena conjecture is that all the important dynamics (at least in an appropriate limit) takes place on the four-dimensional boundary of that space ("holography"). The remaining five dimensions of the superstring, which form a sphere, contribute the SO(6) internal symmetry of the $\mathcal{N} = 4$ superconformal group: including the fermions, this superstring then describes the color singlets of $\mathcal{N} = 4$ super Yang-Mills theory.

In this paper we will find a similar correspondence ("duality") between this same string theory and a different maximally supersymmetric Yang-Mills theory. In the following section we describe our different holography: it is based on a zero-radius limit where AdS_5 shrinks at the center to become a (projective) lightcone. It also emphasizes the boundary, but the fifth dimension remains as a non dynamical (to leading order) auxiliary variable. The construction is illustrated by the case of the particle. (It is a geometric construction that does not require the explicit appearance of a string.) In section 3 we extend this limit to an expansion. It uses the same couplings available in the Maldacena description, but after a rescaling of the coordinates.

The other method we apply is the random worldsheet lattice, introduced in section 4. This approach focuses on fields that do not appear explicitly otherwise: upon identifying the discretized path integral as Feynman diagrams of a matrix field theory, these "constituent" fields of the "composite" string states are naturally identified in a QCD string picture, where $\mathcal{N} = 4$ supersymmetric Yang-Mills is a "simpler" analog of QCD. Their gauge group and coupling are not the same as those of the similar ground states of the open string: in QCD language, "gluons" are not the same as massless (in this idealized model) "ρ mesons".

These two methods are combined in section 5 in application to the bosonic string, which illustrates how the fifth dimension reduces, in leading order, to a kind of Schwinger parameter. The result is that the tree graphs of this model are identical to those of massless (wrong-sign) ϕ^4 theory (in contrast to random lattice quantization about flat space, which produces gaussian propagators). In section 6 this derivation is used to relate the couplings of the closed string, open string (a la Maldacena), and random matrix theory.

In the final section we extend the analysis to the superstring. The limit kills the Wess-Zumino term (as well as shrinking the sphere S^5 to a point). In a (probably unrealistic, but covariant) "long string" gauge, the tree graph Feynman rules are those that follow from a manifestly $\mathcal{N} = 4$ supersymmetric action for a matrix field that we identify with the super Yang-Mills field.

2. Holographies

The use of higher dimensions to manifest conformal symmetry was originally proposed by Dirac [2, 3] in a form different from AdS/CFT: in six dimensions, with appropriate signature, the Lorentz group alone yields the four-dimensional conformal group. Restriction to the lightcone eliminates one coordinate, preserving this group (but breaking the unwanted 6D translational invariance). This 5D lightcone is a limit of 5D Anti-de Sitter space: in terms of 6D coordinates z, the latter is described by $z^2 + R^2 = 0$ for some radius R (giving constant 5D curvature $-1/R^2$), and the former by $z^2 = 0$. The fifth coordinate can then be eliminated by introducing a scale invariance, identifying z with any real multiple of itself; fields on this space are required to be homogeneous in z. The resulting projective lightcone yields a description of conformal field theories equivalent to the usual 4D one, but makes conformal symmetry manifest. (A version of this approach was the basis of the ADHM construction of instantons [4].)

Our approach will be to perturb the string action in R: the lowest-order approximation to Anti-de Sitter space will then be the projective lightcone. Interpreting holography in the more general sense, as describing dynamics with one less spatial dimension, we therefore have a new holographic approach to AdS/CFT (although

JHEP10(2000)040

this difference alone will not yet give our new AdS/CFT correspondence). As for the Maldacena holography, this is not a proof of holography, but rather an expansion about a holographic limit.

In practice, the Maldacena approach is usually applied by first reducing the string to 5D gauged supergravity, i.e. truncating the 10D superstring to a 5D superparticle. For an explicit comparison of the two holographies, we first consider the simpler bosonic analog, the classical mechanics of a conformal scalar particle in a background scalar field. Upon (first-)quantization, this approach relates directly to pure field theory, as a particular set of Feynman tree graphs for ϕ^n theory can be represented in terms of the propagator of a particle in a ϕ^{n-2} background, simply modifying $p^2 \to p^2 + \phi^{n-2}$.

A particle in Anti-de Sitter space AdS_{D+1} for arbitrary dimensions D is then described by

$$S = \int d\tau \ \left[\frac{1}{2}\dot{z}^2 + \frac{1}{2}\lambda(z^2 + R^2) + \hat{\xi}(z) \right] \tag{2.1}$$

with Lagrange multiplier λ and (perhaps composite) background scalar field $\hat{\xi}$. The relation to the usual 4D coordinates is given by

$$x^a = \frac{z^a}{z^+} \,, \qquad x^0 = \frac{1}{z^+} \tag{2.2}$$

in a lightcone basis (z^+, z^-, z^a) with respect to the two extra coordinates. (We also identify $z \to -z$ by restricting $x^0 \geq 0$.) The restriction to the boundary

$$\hat{\xi}(z) = \delta(x^0)\xi(x^a) \tag{2.3}$$

is applied as a limit. Solving the constraint on z from varying λ, the action then reduces to

$$S = \int d\tau \ \left[\frac{1}{2} \frac{(\dot{x}^a)^2 + R^2(\dot{x}^0)^2}{(x^0)^2} + \delta(x^0)\xi(x^a) \right] \tag{2.4}$$

and x^0 remains dynamical (although we can decouple it in the limit $R \to 0$). The result is not the usual 4D scalar.

On the projective lightcone, the lagrangian form of the action is the result of taking $R \to 0$ on the previous:

$$S = \int d\tau \ \left[\frac{1}{2}\dot{z}^2 + \frac{1}{2}\lambda z^2 + \hat{\xi}(z) \right] . \tag{2.5}$$

Projective invariance is incorporated into reparametrization invariance [5]:

$$\delta z = \epsilon \dot{z} - \frac{1}{2}\dot{\epsilon} z \,, \qquad \delta\lambda = \epsilon\dot{\lambda} + 2\dot{\epsilon}\lambda - \frac{1}{2}\dddot{\epsilon} \, . \tag{2.6}$$

This is preserved by the background only if

$$z \cdot \frac{\partial}{\partial z} \, \hat{\xi} = -2\hat{\xi} \, . \tag{2.7}$$

JHEP10(2000)040

This homogeneity condition on the background is solved as

$$\hat{\xi}(z) = (x^0)^2 \xi(x^a) \,. \tag{2.8}$$

Again solving the constraint, the action becomes the usual

$$S = \int d\tau \, \left[\frac{1}{2}(x^0)^{-2}(\dot{x}^a)^2 + (x^0)^2 \xi(x^a) \right], \tag{2.9}$$

where x^0 has not disappeared, but has become the worldline metric. Although non dynamical, it plays an important role upon quantization: it yields the Schwinger parameters. The result is the standard description of a real massless scalar in a background. The background can itself be identified with the usual bosonic field by dimensional analysis: since the background is the usual conformal scalar field theory interaction term less two powers of the scalar, we have

$$\hat{\xi} = \hat{\phi}^{4/(D-2)} \longrightarrow \hat{\phi}(z) = (x^0)^{(D-2)/2} \phi(x^a) \,, \tag{2.10}$$

i.e. $\hat{\xi} = \hat{\phi}^2$ and $\hat{\phi} = x^0 \phi$ in $D = 4$.

3. Expansions

There are two dimensionless couplings in the $AdS_5 \times S^5$ description of type-IIB superstring theory (after including appropriate powers of α'). One is the "string coupling", which appears for any background ("vacuum"), and is associated with the asymptotic value of the dilaton field. It is included in the worldsheet action through a topological term, which we will include implicitly. Both of the expansions we will consider expand in this coupling.

The other is the radius R of AdS_5 and S^5. We have already included it, but it can be moved around through coordinate transformations, which are always allowed in a gravitational theory (i.e. the string). (However, we will only need to redefine x^0, which is not considered "observable", so the worst consequence will be field renormalizations.) In the Feynman rules that we will derive, such rescalings of coordinates are equivalent to rescalings of fields, since propagators (and in general, vertices) in conformal theories are powers of coordinates (or momenta); it is the equivalence between active and passive transformations. However, the coupling expansions are dependent on such redefinitions: for convenience, we always expand in R or $1/R$ while keeping x^a and x^0 fixed, so different expansions can be obtained by R-rescalings of the coordinates.

In particular, the AdS metric we used above, motivated by geometrical considerations, is

$$ds^2 = \frac{(dx^a)^2 + R^2(dx^0)^2}{(x^0)^2} \,. \tag{3.1}$$

It differs from that useful in the Maldacena approach,

$$ds^2 = R^2 \frac{(dx'^a)^2 + (dx'^0)^2}{(x'^0)^2} \tag{3.2}$$

through a rescaling of x^0,

$$x^0 = \frac{x'^0}{R} \tag{3.3}$$

or $x^a = Rx'^a$. Specifically, we expand ds^2 in R^2, x fixed, while the other approach expands in $1/R^2$, x' fixed. The former is a geometrical expansion of AdS about the (projective) lightcone, while the latter is a JWKB expansion, since R^2 appears as an overall factor in the string mechanics action. In the Maldacena coordinates, our expansion corresponds to the limit $R \to 0$, $x'^0 \to 0$, x'^0/R fixed: it is an expansion about the boundary, so the two holographies are closely related. However, in our holography x^0 remains as a coordinate, even after taking the limit (i.e. for the leading term in the expansion): as we saw in the example of the previous section, the limiting theory is not expressed directly as a boundary theory, but as a theory where one spatial dimension is non dynamical, although still playing a useful role.

Note that when we say "Maldacena approach" we begin with $AdS_5 \times S^5$. Maldacena actually began with a different, asymptotically flat space, where the relationship to D-branes (and thus the open string Yang-Mills group) was clear, and took an additional limit to obtain $AdS_5 \times S^5$. For us there is no advantage to starting with the other space. Both are 3-brane solutions to the classical equations, differing by a constant of integration, where asymptotically flat space effectively has an extra 3-brane at the boundary: explicitly, solutions for the various background fields (including the metric) with parallel BPS (small supersymmetry multiplet) 3-branes are characteristically expressed in terms of a function of the form:

$$a + \sum_i \frac{b_i^2}{(y - y_i)^4} \tag{3.4}$$

for some constants a, b_i, y_i, where y are the coordinates "orthogonal" to the branes: y_i are their positions, and b_i their charges. The constant of integration a is generally fixed to 1 for an asymptotically flat boundary, but otherwise is arbitrary, and can also be set to 0. In particular, it can be generated by choosing one (or some) of the branes to be at $y_1 = \infty$ (with $b_1 = \infty$). In the Maldacena case, the remaining y_i's are set to 0. We are thus free to set $a = 0$ from the beginning as a choice of background solution, rather than starting with a less desirable choice and requiring an additional, irrelevant limit. The choice $a = 0$ is difficult to interpret in a conventional S-matrix approach (although not an obstruction in either a β-function or string field theory approach); but it is exactly the unconventional properties (even the definition) of the S-matrix in (super)conformal field theories that is being studied in the AdS/CFT correspondence.

JHEP10(2000)040

So far we have excluded the α' dependence of the action: after the same kind of redefinitions we have made for R, it always appears in the combination R^2/α' in $AdS_5 \times S^5$. In any case, it can always be restored by the substitution $R^2 \to R^2/\alpha'$. (In the particle case any such overall factor in the spacetime metric can be absorbed into the worldline metric because of lack of worldline scale invariance.) We will thus always refer to R^2 as either the radius (squared) or the tension (although it is a little more like a torque). This interpretation is analogous to that of $\hbar$ and the coupling g in any gauge field theory: there the dimensionless combination is $\hbar g^2$, and the two appear only in this combination, after appropriate field redefinitions. Expansion in this $\hbar$ produces the usual loop expansion of field theory. However, we can redefine $x \to x/\hbar$ and $\hbar g^2 \to g^2/\hbar$; then expansion in $\hbar$ produces the JWKB expansion of (relativistic) quantum mechanics [6]. Introducing $\hbar$ into field theory defines the dimension of "mass" independently from "length", but necessarily in a trivial way if the theory is scale invariant. Thus, we could write

$$ds^2 = \frac{1}{\alpha'}\frac{(dx^a)^2 + R'^2(dx^0)^2}{(x^0)^2}, \qquad R^2 \equiv \frac{R'^2}{\alpha'}. \tag{3.5}$$

Then the Maldacena interpretation comes from setting $R' = 1$ and expanding in α', while our interpretation sets $\alpha' = 1$ and expands in R'. In either case R remains as a non-trivial dimensionless coupling.

4. Random lattices

Besides the R expansion, which leads to the projective lightcone, the other main ingredient in our analysis is the random worldsheet lattice [7]. Although either of these two methods can be applied separately to the string on $AdS_5 \times S^5$, we will find that the combination solves some mutual problems. The random lattice approach is based on introducing a lattice regularization for the worldsheet coordinates. Since the worldsheet is curved, the lattice is irregular, and the sum over geometries in the path integral becomes a sum over lattices of different geometries. By identifying the lattices as Feynman diagrams, the first-quantized path integral for the string becomes a sum of Feynman diagrams for some *particle* field theory, whose propagators and vertices are read from the discretized action of the string. This discretization is used in conjunction [8] with the $1/N$ expansion [9], which allows a 2D topology to be associated with any Feynman diagram by using fields that are $N \times N$ matrices. String states are identified with singlets under the corresponding $\mathrm{U}(N)$ symmetry: in a QCD interpretation, the Feynman rules are for a chromodynamic field theory, describing gluons and quarks, while the string describes hadrons. The motivation for such a dual approach to strong interactions is that one could continue to use perturbative QCD to describe processes with large transverse momenta while applying string theory to describe the asymptotic spectrum and Regge behavior: two different practical perturbation expansions for the same theory, each accurate in its region of momentum space.

Thus the random lattice approach leads to a different interpretation of the various Yang-Mills fields than that used in the Maldacena approach. In the hadronic interpretation of string theory, open strings describe mesons, while closed strings describe "pomerons". In QCD strings and their generalizations (super Yang-Mills strings, etc.), the mesons are identified as quark-antiquark states bound by gluons, while the pomerons are identified as "glueballs". The random lattice approach explicitly associates a QCD-like particle theory of "partons" (gluons and quarks) with a hadron-like string theory of mesons and pomerons. Previously little explicit success has been obtained with this approach except for the bosonic string, using scalar partons. Here we propose, through an AdS/CFT approach, to identify maximally supersymmetric Yang-Mills theory with the partons, as the only known maximally superconformal theory with fields than can be identified as matrices with respect to an internal symmetry. Of course, in this (unbroken) supersymmetric model, fermions come in unphysical group representations, and the lightest hadrons appear as massless (as in the original Maldacena model). Thus, while the partons include massless gluons, the open string states include massless ρ mesons. Both the gluons and the ρ's appear in supersymmetric multiplets, and both are associated with gauge fields. (Originally, Yang and Mills identified non-abelian gauge fields as massless ρ's. After the Higgs effect was discovered, but before QCD, phenomenological models of low-mass hadrons made the ρ's massive by eating the scalars of linear or non-linear σ-models.) This differs from the Maldacena interpretation, where the massless open-string states are themselves identified with gluons, while the closed string states are still identified with pomerons (glueballs). Our interpretation thus more closely coincides with the original QCD interpretation, where the gluons are not themselves strings. (However, as an alternative interpretation of our approach, the states of the underlying particle theory could be identified instead as "preons".) Independent of interpretation, the random lattice approach necessarily involves two different groups, as the group acting on the matrices of the underlying particle theory is not necessarily identical to the group acting on the open string states. More importantly, our AdS/CFT correspondence necessarily differs from that of Maldacena because the conformal field theory we analyze is not that of the open string states, but that of the underlying particle theory. As a consequence, unlike the Maldacena approach, in our AdS/CFT correspondence the Yang-Mills fields that we identify with the gluons appear explicitly, so in principle both perturbative and non-perturbative factors of QCD amplitudes can be examined together. Furthermore, in the D-brane picture of the open strings, where a closed string can create an open string pair (as in type-I theory, except here for type II the ends of the open strings are confined to the D3-branes), the random lattice treats both the open and closed strings as flux tubes of the partonic gluons: as in the 't Hooft picture for QCD, a closed string (pomeron/glueball) can still be considered a bound state of two open strings (mesons), but both types of strings are coherent states of arbitrary numbers of gluons; the glueballs are really closed flux tubes, while the open strings are open flux tubes.

One problem with the random lattice approach is that quadratic string actions give gaussian propagators: discretizing the usual $(\partial x)^2$ term common to all strings,

$$\int Dx \; e^{-S} \sim \int \left(\prod_I dx_I \right) e^{-\frac{1}{2}\sum_{\langle IJ \rangle}(x_I - x_J)^2} \tag{4.1}$$

for vertices I with links (propagators) $\langle IJ \rangle$. (Other terms in superstring actions yield vertex factors, and "numerators" for the propagators [10].) Consequences are non-parton-like behavior at large transverse momenta [11], and essentially no degrees of freedom past the Hagedorn temperature [12] (since the propagator has no poles), in what should be the parton plasma phase. One proposal was to introduce Schwinger parameters as a new degree of freedom in the string action [13], but no correspondence with the usual strings was obtained. However, the example of section 1 suggests a more natural way to incorporate the extra variable within the conventional string framework. (The propagators in the Maldacena approach are not the usual ones, nor gaussians, but require Bessel functions.)

5. Bosonic string

The analysis of section 1 generalizes straightforwardly (except for reparametrizations of the worldline/sheet) to (five dimensions of) the bosonic string: since we use the same background spacetime metric, we need only replace derivatives and integrals with worldsheet ones, $d\tau \to d\tau d\sigma$, $\dot{f}^2 \to (\partial f)^2$, where the former implicitly includes the worldsheet measure $\sqrt{-g}$, and the latter the worldsheet metric as $g^{mn}(\partial_m f)(\partial_n f)$. (We consider here AdS_5 and ignore effects of worldsheet and spacetime conformal anomalies.) Then introducing the random lattice as in the previous section, the action

$$\begin{aligned} S &= \int d^2\sigma \, \sqrt{-g} \, \left[\frac{1}{2} g^{mn}(\partial_m z)(\partial_n z) + \frac{1}{2}\lambda(z^2 + R^2) \right] \\ &\longrightarrow \sum_{\langle IJ \rangle} \frac{1}{2}(z_I - z_J)^2 + \sum_I \frac{1}{2}\lambda_I(z_I^2 + R^2) \end{aligned} \tag{5.1}$$

is path integrated, with insertions of background sources ("punctures") $\zeta(z)$, as

$$\mathcal{A} = \int DzD\lambda \; e^{-S}\zeta \cdots \zeta \longrightarrow \int \left(\prod_I dz_I d\lambda_I \right) e^{-S}\zeta \cdots \zeta \,. \tag{5.2}$$

Integrating out λ,

$$\mathcal{A} = \int \left[\prod_I dz_I \delta(z_I^2 + R^2) \right] e^{-\frac{1}{2}\sum_{\langle IJ \rangle}(z_I - z_J)^2} \prod_{I'} \zeta(z_{I'}) \,, \tag{5.3}$$

where I' are just the points where the background is attached. (For simplicity, we have introduced pointlike sources, corresponding to ground states, e.g. $e^{ik\cdot x}$. Sources for n-th derivatives use functions of n adjacent points, as $\partial x \to x_I - x_J$ in the action.)

Note that we have not determined the R dependence of the measure; it can easily be fixed at the end, as we will describe in the following section. We have also used the simplest discretization of the action: it actually corresponds not to the true distance squared, but the distance as would be measured in the six-dimensional embedding space. Lattice actions are always ambiguous to such "higher-derivative" terms; our choice is not only simpler than the true distance, but single-valued at the antipode in the case of the sphere. (In that case, we have replaced the distance s with $2\sin(s/2)$.)

In the random lattice approach, 't Hooft's analysis of the $1/N$ expansion is used to identify 2D topology of Feynman diagrams: the double lines representing the gluon propagators give the gluons a slight "stringiness", so that the leading order in $1/N$ is identified pictorially with a diagram that is planar with respect to the double lines. The double lines are a consequence of the use of a matrix representation, and can be applied to any theory whose fields are matrices. The value of the Euler number χ of an arbitrary diagram can be identified unambiguously, and contributes a factor N^χ, with all remaining N-dependence occuring together with the defining-representation coupling g^2 (appropriate to the double-line notation) only in the combination of the adjoint-representation coupling Ng^2. Since explicit color lines do not appear until the introduction of the double lines, the sources appearing in the path integral above are necessarily color singlets. However, the Feynman rules for color non singlets, and in particular the matrix fields themselves, are obvious from the rules for singlets.

Again identifying the discretized first-quantized string path integral as a Feynman diagram, and introducing the double lines, we read off the Feynman rules as

$$\begin{aligned} \text{Propagator:} \quad & \delta_{i'}^{j} \delta_{i}^{j'} e^{-(z-z')^2/2} \,, \\ \text{Vertex:} \quad & \delta_{j_2}^{i_1} \delta_{j_3}^{i_2} \cdots \delta_{j_1}^{i_n} \int d^6 z \; \delta(z^2 + R^2) \,, \\ \text{External line:} \quad & \hat{\phi}_i{}^j(z) \,. \end{aligned} \tag{5.4}$$

This external line is for a background matrix field $\hat{\phi}$ itself; the color singlet sources considered above are traces of powers of such fields. We have associated the world-sheet with Feynman diagrams with n-point vertices; (spacetime) conformal invariance will restrict this choice.

To compare with the usual 4D rules, we replace $z \to (x, x^0)$ as before. We find

$$(z - z')^2 = \frac{(x - x')^2 + R^2(x^0 - x'^0)^2}{x^0 x'^0}\,. \tag{5.5}$$

For the projective lightcone limit $R \to 0$, we also replace $\hat{\phi} = x^0\phi(x)$. The rules then become, now dropping indices,

$$\begin{aligned} \text{Propagator:}\quad & e^{-(x-x')^2/2x^0x'^0}\,, \\ \text{Vertex:}\quad & \int d^4x\; dx^0 (x^0)^{-5}\,, \\ \text{External line:}\quad & x^0\phi(x)\,. \end{aligned} \tag{5.6}$$

Because of the simple x^0 dependence in the projective lightcone limit, this coordinate can easily be integrated out explicitly in tree graphs. The easiest way is to examine the Schwinger-Dyson equation (what would be the integrated field equation): choosing as random lattices the diagrams of ϕ^4 theory (i.e. 4-point vertices only),

$$\hat{\phi}(z) = \int d^6z'\; \delta(z'^2) e^{-(z-z')^2/2} \hat{\phi}^3(z') \tag{5.7}$$

which is now

$$x^0\phi(x) = \int d^4x' dx'^0 (x'^0)^{-5} e^{-(x-x')^2/2x^0x'^0} (x'^0)^3 \phi^3(x')\,. \tag{5.8}$$

Integrating $\int_0^\infty dx'^0$,

$$\phi(x) = \int d^4x' \frac{1}{\frac{1}{2}(x - x')^2} \phi^3(x')\,. \tag{5.9}$$

This is the usual Schwinger-Dyson equation for massless ϕ^4 theory, as x^{-2} is the usual massless propagator. (Of course, the integrals can be evaluated as easily in momentum space, or for conformal scalars in other dimensions. If we like, we can also add a source term as $\hat{\phi}^3 \to \hat{\phi}^3 + \hat{J}$, or $\phi^3 \to \phi^3 + J$, as $\hat{J} = (x^0)^3 J(x)$.) Thus the role of x^0 in the projective lightcone expansion has been reduced effectively to that of Schwinger parameters. (Extra dimensions act similarly to Schwinger and Feynman parameters in the Parisi-Sourlas formalism also [14], particularly when applied to covariantizing the usual lightcone [15].)

The final Feynman rules for tree graphs are therefore the usual ones of massless ϕ^4 theory:

$$\begin{aligned} \text{Propagator:}\quad & \frac{1}{\frac{1}{2}(x - x')^2}\,, \\ \text{Vertex:}\quad & \int d^4x\,, \\ \text{External line:}\quad & \phi(x)\,. \end{aligned} \tag{5.10}$$

(Adler [3] also formulated Feynman rules for conformal theories on the projective lightcone, but he set $x^0 = 1$ by hand as a gauge condition, which is not applicable here.) Note that the usual massless propagator resulted only because we assumed the usual conformal ϕ^4 interaction: if we had assumed a different power of $\hat{\phi}$ (with x^0 dependence consistent with conformal invariance), a different power of x^2 would have appeared, consistent with the modified scale weight of ϕ, but implying a non-local kinetic operator.

We can see the above procedure implemented explicitly in any tree graph, integrating one vertex at a time. Unfortunately, the final integration in any diagram will always produce an extra factor

$$\int_0^\infty \frac{dx^0}{x^0} = \infty \,. \tag{5.11}$$

This factor appears with each connected graph; it accompanies the usual energy-momentum conservation δ-function

$$\delta\left(\sum p\right) \sim \int d^4x = \infty \tag{5.12}$$

when conservation is enforced (e.g. when squaring for probabilities), that is associated with the volume of spacetime. The new infinity can thus be associated with conservation of the dilatation (scale) charge. It may be possible to absorb it into some renormalizations.

In loop diagrams the x^0 integration is more complicated, presumably because of contributions corresponding to "bulk modes". Such contributions should be compared to similar corrections coming from higher orders in R.

6. Dualities

In the usual flat background there are two different ("dual") obvious particle limits for strings: $\alpha' \to 0$ and $\alpha' \to \infty$. In both these limits (expansions) one can still also expand in the string coupling g_s. The infinite tension ($\sim 1/\alpha'$) limit ($\alpha' \to 0$, or "zero slope") describes the ground state of the string by shortening it to a point (particle). In the QCD string picture, this particle from the open string is the ρ meson (or other light mesons). The flavor coupling constant is identified with the open string coupling $\sqrt{g_s}$ (up to any power of α' required to give it the right engineering dimensions). (The limit $\alpha' \to 0$ can also give the classical mechanics limit of the string, depending on how σ is scaled.)

On the other hand, in the zero tension limit ($\alpha' \to \infty$) the string "falls apart": there is no tension to hold the pieces together. In the QCD string picture, this is the limit where the gluons decouple. Clearly the tension is related to the color coupling constant. Meanwhile, the string coupling is identified with $1/N$ according

JHEP10(2000)040

to the 't Hooft analysis, in terms of the number N of colors. Unfortunately, the usual strings don't resemble QCD strings very closely, at least not when expanded about flat space. We now examine how these naive relations are affected when expanding about $AdS_5 \times S^5$.

We begin by analyzing how the two (dimensionless) coupling constants appear in both the string and matrix field actions. The worldsheet path integral can consist of (1) the "propagator" ($(\partial x)^2$) term, (2) a Wess-Zumino term, (3) the (topological) worldsheet curvature term, (4) a (worldsheet) cosmological constant term, and (5) the measure. There is no Wess-Zumino term for the bosonic string; in any case, such a term for superstrings introduces couplings only as higher-derivative corrections to the leading interaction terms [10].

The curvature term is responsible for the topological $1/N$ coupling already considered: it appears in the string action S, contributing to the path integral as e^{-S}, as a term

$$S_\chi = \chi \, \ln\left(\frac{1}{N}\right) = \chi \, \ln g_s \Longrightarrow g_s = \frac{1}{N} , \tag{6.1}$$

where the Euler number χ is expressed as the integral of the curvature. Thus, in this approach the closed string coupling is simply $1/N$ (and $1/\sqrt{N}$ for the open string), according to the general result of 't Hooft. This differs from the result of conventional random lattice models because we make our analysis for the critical dimension, where presumably there are no renormalization effects associated with a worldsheet conformal anomaly. (We have ignored the extra compactification dimensions so far, but will return to them below, and use the correct counting of dimensions for the $D = 10$ superstring.)

The cosmological term is equivalent to the measure:

$$\prod_I dx_I \, e^{\sum_I \ln \, \mu_I} = \prod_I dx_I \, \mu_I \, . \tag{6.2}$$

Therefore, for convenience we associate such non-derivative terms with the measure by definition. Again assuming no renormalization effects because of the critical dimensionality, the measure is then determined by the same methods used in nonrelativstic quantum mechanics: the exact measure for path integrals can be derived by a lattice definition of path integration in terms of ordinary integrals (as we do here) by starting from a hamiltonian form of the action, which follows directly from the operator formalism. Then there is no measure to start (other than the usual $\sqrt{2\pi}$'s), but a measure is generated by integrating out the momenta. The result is a path integral of the form

$$\prod_I \left(d^D x_I \, \sqrt{G} \right) \exp \left[-\frac{1}{2} \sum_{\langle IJ \rangle} G_{mn} (x_I^m - x_J^m)(x_I^n - x_J^n) \right] , \tag{6.3}$$

(where $G = \det(G_{mn})$). This is clearly covariant in the continuum limit, since $d^D x\sqrt{G}$ is the usual covariant integration measure. There is some ambiguity in the discretized version (e.g. $G(x_I)$ vs. $G(x_J)$) related to higher-derivative corrections to the action; we have made a choice consistent with the (worldsheet) continuum limit, and with conformal invariance before the limit. However, this expression for the measure is exact including constant factors (once a $(2\pi)^{-D/2}$ is included with each $d^D x$), which have no effect on covariance. (For example, the factor $(m/\hbar)^{D/2}$ appears for the free non-relativistic particle, with "metric" $(dx)^2 m/\hbar$.) These factors are required to define the continuum limit: for example, a spurious factor of 2 at each vertex would generate a 2^V for V vertices, which would become infinite in the continuum limit $V \to \infty$. In our case the existence of such a limit is equivalent to worldsheet Weyl scale invariance: integrating out half the vertices should produce a result of the same form as the original, except for modification of the exponent consistent with doubling the worldsheet area per vertex. (The intrinsic 2D "area" of the worldsheet lattice is the number of vertices.)

In our case the R dependence for the bosonic string, before taking the projective lightcone limit, is given by a measure factor of R accompanying the dx^0, following from the term $(Rdx^0/x^0)^2$ in the string action. Compactified dimensions (such as for S^5) do not contribute: their measure must be normalized to give 1 upon integration, since in the projective lightcone limit they do not contribute to the string action; i.e. $(\int d\Omega_5)^V = 1^V \to 1$ as $V \to \infty$. ($R \to 0$ shrinks them to a point. Before taking the limit their measure is more complicated: in the above derivation the momenta are quantized; the sum is not as simple as a gaussian integral, and produces a more complicated measure factor.)

In the projective lightcone Feynman rules of the previous section, we can then associate an R with each vertex, with no extra factors from the propagator (since the string action has no R dependence in the projective lightcone limit). This corresponds to a matrix field lagrangian

$$L \longrightarrow N\,\mathrm{tr}\left(-\frac{1}{2}\phi\Box\phi - R\frac{1}{4}\phi^4\right). \tag{6.4}$$

(The coupling is negative so the vertex is positive: the bosonic functional integral is always positive, being the integral of an exponential.) We thus have the identifications

$$g_s \sim \frac{1}{N_c}\,, \qquad R \sim N_c\, g_c^2\,, \tag{6.5}$$

where N_c is the rank of the "color" gauge group of the gluon and g_c its coupling, while g_s is the closed string (pomeron) coupling. This result agrees with the naive qualitative relation between tension (R^2) and color coupling.

The Maldacena approach uses the other duality: the particles of its superconformal Yang-Mills theory are the "massless" ground states of the open superstring. Thus the perturbation expansion used is different: it corresponds neither to Regge nor parton behavior. In that approach the identification between (closed) string couplings and Yang-Mills ones is

$$g_s \sim g_f^2 \,, \qquad R^4 \sim N_f g_f^2 \,, \tag{6.6}$$

where g_f is the open string (gauge meson) coupling, and N_f is the rank of the meson ("flavor") gauge group. The resulting relations

$$\frac{1}{N_c} \sim g_f^2 \,, \qquad N_c^4\, g_c^8 \sim N_f g_f^2 \tag{6.7}$$

suggest a new type of duality between SYM theories. (It is similar to Seiberg's duality for $N = 1$ theories [16] in the sense that the rank of the group is changed together with the coupling constant.)

7. Superstring

The classical $AdS_5 \times S_5$ superstring action is [17]

$$S = \int d^2\sigma \left[\frac{1}{2}\sqrt{-g} g^{mn} L_m{}^{\mathcal{A}}(1) L_{n\mathcal{A}}(1) + i\epsilon^{mn} \int_0^1 ds\; L_m{}^{\mathcal{A}}(s) \eta^{ij} \bar{\Theta}^i \gamma_{\mathcal{A}} L_n{}^j(s) \right] , \tag{7.1}$$

where

$$\begin{aligned} d\sigma^m L_m{}^i(s) &= \left(\frac{\sinh(sM)}{M} D\Theta \right)^i , \\ d\sigma^m L_m{}^{\mathcal{A}}(s) &= e_{\mathcal{M}}{}^{\mathcal{A}}(X) dX^{\mathcal{M}} - 2i\bar{\Theta}^i \gamma^{\mathcal{A}} \left(\frac{\sinh^2(\frac{1}{2}sM)}{M^2} D\Theta \right)^i \end{aligned} \tag{7.2}$$

and M is a matrix bilinear in fermions, and $D\Theta$ a covariant derivative on the fermions. Here $\eta^{ij} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\mathcal{A}, \mathcal{M}$ are 10D vector indices. In the (4D) covariant κ gauge [18]

$$\Theta^1 = \gamma_{0123} \Theta^2 \,, \qquad (\gamma_{0123} = i\gamma_4) \tag{7.3}$$

we obtain the gauge fixed action [19, 18] (in terms of the remainder θ of Θ)

$$\begin{aligned} S = \int d^2\sigma \Bigg\{ & \frac{1}{2}\sqrt{-g} g^{mn} \left[\frac{\Pi_m{}^a \Pi_{ma} + R^2 (\partial_m x^0)(\partial_n x^0)}{(x^0)^2} + R^2 (d\hat{y})^2 \right] - \\ & - i\epsilon^{mn} \frac{R(\partial_m \bar{\theta})\gamma \cdot \hat{y} (\partial_n \theta)}{x^0} \Bigg\} \end{aligned} \tag{7.4}$$

in terms of the unit 6-vector $\hat{y}$ (of the internal SO(6)) on S^5 and the 4D supersymmetric differential

$$\Pi_m{}^a = \partial_m x^a - i\bar{\theta}\gamma^a \partial_m \theta \,, \tag{7.5}$$

where we have reinstated the dependence on R as in section 2. Remember that the usual AdS/CFT correspondence uses the JWKB expansion. Our new correspondence uses instead an expansion in R, the leading term of which is the projective lightcone limit ($R \to 0$), which leaves only the first term,

$$S = \int d^2\sigma \, \frac{1}{2}\sqrt{-g}g^{mn}\frac{\Pi_m{}^a\Pi_{ma}}{(x^0)^2} \,. \tag{7.6}$$

The covariant gauge is more appropriate for expanding around "long string" configurations [18], for which one can use the static gauge $x^0 = \sigma^0$, $x^1 = \sigma^1$ to obtain a non-degenerate fermionic kinetic operator $\mathcal{A}$ ($\mathcal{A}^2 \neq 0$). (e.g. it was used in [20] to compute quantum corrections to the gauge theory $q\bar{q}$ potential via the AdS/CFT correspondence.) For σ-independent x-configurations ("short strings"), one can instead use the light-cone gauge $(\gamma^3 - \gamma^0)\Theta^I = 0$ [21], resulting in a more complicated action (but with a non-degenerate fermionic kinetic operator). Here we will ignore such subtleties; our approach can be applied for any κ gauge, and will suffer from the same difficulties, but we choose the covariant gauge to illustrate how this covariance is reflected in the form of the Feynman rules.

The projective lightcone superstring action is the same as the bosonic one except for the supersymmetrization of the 4D differential $\partial_m x^a \to \Pi_m{}^a$. The discretized form is

$$x_I^a - x_J^a \longrightarrow x_I^a - x_J^a - i\bar{\theta}_I\gamma^m\theta_J \,. \tag{7.7}$$

The Feynman rules then follow directly from the bosonic case: as there, x^0 dependence is determined by requiring that the propagator, vertex, and external line factor be dimensionless. (The dimensions of θ are $[\theta] = [x^{1/2}]$.) The result for a ϕ^n interaction is then

$$\begin{aligned} \text{Propagator:} \quad & d^{16}e^{-(x-x')^2/2x^0x'^0}\delta^{16}(\theta-\theta') \,, \\ \text{Vertex:} \quad & \int d^4x \; d^{16}\theta \; dx^0(x^0)^3 \,, \\ \text{External line:} \quad & (x^0)^{-4/n}\phi(x,\theta) \,. \end{aligned} \tag{7.8}$$

Here we have used the identity

$$f(x - x' - i\bar{\theta}\gamma\theta') = d^{16}f(x-x')\delta^{16}(\theta-\theta') \tag{7.9}$$

for any function f, where d is the covariant spinor derivative and d^{16} the antisymmetric product of all its components. (This can be derived, e.g. by writing $d^{16} = \int d^{16}\zeta \; \exp(\zeta d)$.)

Integrating out x^0 in tree graphs as before, the tree rules become

$$\begin{aligned} \text{Propagator:} \quad & d^{16}\delta^{16}(\theta-\theta')\left[\frac{1}{2}(x-x')^2\right]^{4/n} \,, \\ \text{Vertex:} \quad & \int d^4x \; d^{16}\theta \,, \\ \text{External line:} \quad & \phi(x,\theta) \,. \end{aligned} \tag{7.10}$$

(Certain values of n give divergent numerical factors, which we remove by "renormalization".) These imply the classical action

$$S = N\,\mathrm{tr}\int d^4x\; d^{16}\theta\; \left(\frac{1}{2}\phi d^{16}\Box^{4/n-6}\phi + R\frac{1}{n}\phi^n\right), \tag{7.11}$$

where we now include the coupling, which arises in the same way as in the bosonic case. Here we have used the identity

$$(d^{16})^2 \sim \Box^8 \Longrightarrow (d^{16})^{-1} \sim \frac{d^{16}}{\Box^8}. \tag{7.12}$$

(This is easy to check in momentum space by choosing the lightcone frame for the case $p^2 = 0$ and the rest frame otherwise. In the former case half the d's have vanishing squares; in the latter the d's are proportional to γ matrices for SO(16), and thus d^{16} to "γ_5".)

Unfortunately this action is non local. Its form is dictated by scale invariance, once the interaction and d^{16} numerator are assumed. The non locality is probably due to the improper gauge fixing of κ symmetry, which however preserves supersymmetry and scale invariance. We expect a better understanding of these issues in the string action will lead to better particle field theory actions, as illustrated by the bosonic AdS_5 string example, since the formalism (1) allows explicit elimination of x^0, (2) produces propagators that are powers of x^2 (or p^2), and (3) preserves the global symmetries of the string action. (Serious modifications are expected in order to avoid the no-go theorem for maximally supersymmetric theories [22].)

For any covariant gauge, our approach produces manifestly $\mathcal{N}$=4 supersymmetric supergraphs and (gauge-fixed) actions. Because of the non-abelian gauge group, these would most likely be identified with the lowest-spin $\mathcal{N}$=4 multiplet, super Yang-Mills. After a clarification of the above issues, this approach should give a better understanding of the elusive $\mathcal{N}$=4 superspace formulation of this theory.

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